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Instability and bifurcation in a price formation model

Analytical and numerical analysis

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Abstract

Following M.d.M.González, M.P.Gualdani and J.Solà-Morales' work on J.-M. Lasry and P.L. Lions' evolutive model, the main goal of this work is to understand and analyze how a Hopf Bifurcation may or may not modify the behaviour of a predictive model using both numerical and analytical techniques.
A numerical simulation will be performed to actually determine the kind of bifurcation and its properties, proving moreover a Hopf characterization claim for the predictive model.

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Introduction and summary of results

At the begining of the twenty-first century, the ancient technique of trading was under a technological conversion. Pricing meant much more than *strategically exchanging things or even money*, the complexity of such a process went beyond expectations.

In this project we link this massive economical remake with stability and bifurcation theory, studying how instabilities under a pricing approximation may be appropiate when trying to emulate the behaviour of a price. The aim of this work is to study the pricing model first designed by Jean-Michel Lasry and Pierre-Louis Lions [2], and as well to work around the modifications suggested by M.d.M.González, M.P.Gualdani and J.Solà-Morales [1], and understand and calculate both analytically and numerically how a Hopf Bifurcation comes into play in this model.

The idea is to first define a model to work with: This new model [1], in comparison with the first model [2], comes from the idea that states that a price can not remain stable even if the time spectrum is wide, this leads to the encouragement of creating instabilities using several nonlinear terms in the model.

To determine how these instabilities occur we arranged an analytical inspection and a numerical simulation of the process. Having in mind that a Hopf Bifurcation may occur, the analytical work that is developed in Chapter 3, is focused around bifurcation theory and stability theorems which will make the assumptions left in [1] to seem more clear.

The numerical part of the work, written in Chapter 4, is an experimental view of the problem, providing interesting results about the said model and clearing out questions and assumptions. It is important to declare that this project faces both numerical and analytical problems giving a sufficiently wide vision of the study of such a practical mathematical work.

The main goal is to prove that the claim suggested in [1] is valid; once the results are presented one can say that the goal has been cleared, proving that the instabilities are indeed caused by a Hopf Bifurcation and furthermore, giving details and properties about it.

To conclude, once the problem is clear there are still open doors to posterior studies. We leave an open question about the relationship between the chosen nonlinearities and whether the Hopf Bifurcation is supercritical or subcritical. And even more, we wonder about the accuracy of the model and ask if a simulation of ours resembles a real pricing record, maybe giving way to a machine learning project around this model.

Chapter 1

The Stock Market

1.1 Assets, shares and derivatives

Trading, the most ancient kind of commerce, is the mother of all the financial markets as we know them today, and in fact if one analyzes a trading system of any kind, even the most complex ones, the basis will always resemble the very first kind of trading: a group of people exchanging goods. These goods, known as assets, can nowadays be of almost any kind, from foreign currencies, to financial derivatives, to agricultural products. And in every trading system planned around these goods, there exists a variation of values for the price of that asset.

It is the price of the asset what is typically the focus of every investor, so it has historically been largely studied and analyzed with the aim of developing techniques and models that are built to predict the behaviour of the said price. Typically, to study that behaviour, the evolution of the price of an asset is pictured in a two-dimensional chart, as in Figure 1.1, and just by doing so, some important features can be seen.

Fig. 1.1 AAPL price evolution for a year. NASDAQ, 3 October 2016

Some of the basic properties of a chart or of a section of it may be the mean, the trend, the maximum and minimum and the variance or standard deviation. These words were first included into the financial vocabulary in the early 1900s, being the Portfolio Theory developed by Henry Markowitz the first important result involving the study of these terms [10].

This Modern Portfolio Theory led to the first mathematical model that involved stock trading: The main purpose of it was, given a finite number of shares, to find the combination that lowered the risk returning the maximum expected gain. Even though it was a good way to predict benefits, the process itself wasn't a good way to find the behaviour of a share since the tools used in this technique were applications of multivariate statistics such as discrete time-series that would supply little to none information about the most immediate time. That is the door to the other branch of Mathematical Finance: Derivatives Pricing.

1.2 Where PDEs meet finance: Derivatives Pricing

1.2.1 From binomial trees to Black-Scholes

As we saw in Figure 1.1, the price of a stock may vary following a certain trend, maybe keeping around a mean value, or sometimes hitting a maximum value, but what seems actually clear is that the line forms a non-smooth form, remembering maybe a fractal structure. This naive idea gives rise to adding a stochastic term to the mix, providing a random behaviour to the model. This random behaviour might, for example, be brought by a random walk.

Without wanting to go deeper on this subject, we should define what a random walk is, and in fact, the best way to picture it according to the most basic trading idea is thinking about a binomial tree. The price of an asset might go up or down with the same probability $p = 0.5$, that means that a basic formula for the price can be:

$$S_{t+\Delta t} = \begin{cases} S_t exp(\sigma \Delta t) & \text{if the price grows,} \\ S_t exp(-\sigma \Delta t) & \text{otherwise} \end{cases} \tag{1.1}$$

Where σ is the given asset variance and Δt is a fraction of the discrete time. Considering that S_0 is the price at time 0 and recursively going back in the discrete time one gets the following result:

$$S_t = S_0 exp(\sigma(2X_t - t)) \tag{1.2}$$

Where X_t is a binomial random variable with mean $t/2$ and variance $t/4$ that estimates the number of times the price goes up. Hence, the variable $\frac{(2X_t - t)}{\sqrt{t}}$ follows a normal distribution of mean 0 and variance 1. Thus, using the central limit theorem one gets the first approach in this article to the Black-Scholes model [6]:

$$S_t = S_0 exp(\sigma\sqrt{t}Z - \frac{\sigma^2}{2}t), \quad Z \sim N(0,1) \tag{1.3}$$

1.2.2 Modeling with differential equations: SDEs

Since the Black-Scholes model involves a great area of studies and diverges quite a bit from this paper's subject, only a brief description of the idea of the model and some of its components will be developed, avoiding distraction and going straight to the point that matters: modeling a price-formation process.

Formula 1.3 was built under a very comfortable assumption, that the market did not have a growth rate, and that the interest rate was null, but that is not a case one can find in the real trading market. That is why some more complex models have been developed, being the Black-Scholes one of the most famous.

The Black-Scholes model, as we saw on the previous section, comes from the most primitive idea that prices can only go up and down with a certain probability. To model this phenomena, one adds a Stochastic random variable to the equation providing the random walk as wanted. This model ends up being a differential equation, and one can wonder how a stochastic variable can be treated in this kind of calculus. The Itô integral was the main tool to actually get to the final Black-Scholes equations, being able to treat stochastic variables properly so the final formula is in fact a differentiable equation.

Stochastic Differential Equations (from now on SDEs) defined the breaking point when talking about modeling finantial products providing the idea that differential equations were a good tool to approximate stock market characteristics. Not only SDEs can properly mimic the behaviour of an asset, but actual PDEs have seem to get very good results when it comes to predict a price or a tendency. That is why in the most recent history the trading business has been asking for experts in PDEs and applying this new idea of modelling to some analysis tools and operations. The idea of adding PDEs to financial modelling is born, once again, from the binomial tree concept.

If one assumes $S(R,t)$ as the probability of reaching a price R (height of the branch) in time t (length of the branch), and allows that the probability of going up or down is always $\frac{1}{2}$, we can admit

$$S(t+\Delta t,R) = \frac{1}{2}S(t,R+\Delta R) + \frac{1}{2}S(t,R-\Delta R), \qquad (\Delta R)^2 = \Delta t \tag{1.4}$$

Thus, operating a bit one gets to

$$\frac{S(t+\Delta t,R) - S(t,R)}{2\Delta t} = \frac{1}{4}\frac{S(t,R+\Delta R) + S(t,R-\Delta R) - 2S(t,R)}{(\Delta R)^2} \tag{1.5}$$

moreover, applying limits over the Δ variables and taking into account that as we stated $\sigma = \frac{1}{4}$ one gets

$$\frac{dS}{dt} = \sigma\frac{d^2S}{dR^2} \tag{1.6}$$

That, as we wanted, is a PDE modeling a simple pricing phenomena, and as a matter of fact resembles pretty much the very popular Heat Equation.

On the next section some models will be discussed, and even though there have been plenty of positive uses of applied PDEs in economics, the risks of these models will be explained as well.

1.3 Refashioning the business

It was not until the early seventees of the last century when the theoretical pricing and risk-predictive models were actually put into practice even though there is plenty of evidence that the first written model was defined whithin the first decade of the twentieth century. The real complexity about applying those models was that finding a fitting model was highly incompatible with finding an appliable model, that was due to the high amount of variables that held the most accurate models.

Being inclined to think that the main difficulty about applying models was actually the high amount of properties they held, the thought that simpler models were better began to grow, as some great and "simple" techniques such as the stochastic version of Black-Scholes, its "heat equation transformation" or some portfolio optimisation began to become more and more popular.

As soon as these models were developed enough to actually be applied, it became clear that they were indeed effective, so the biggest banking companies became to use them in a bigger scale. This, accompanied with the automatisation of buying and selling, led to a massive benefit growth concerning those who were able to work with the models. Everything seemed great until the first unexpected crash happened. Those models could not handle critical situations such as *Flash Crashes*, and that is what happened in 1987, when quick changes in the stock market first caused by overvaluation and *Market Psychology* made the models to become useless for a while, making mindless decisions and automatically accepting trading operations that would led to the well known Black Monday of 1987.

For the first time, those brand new models were highly criticised therefore more strict regulation and a deeper knowledge about them was absolutely required. Between all this skepticism, the author would like to rescue these words from Professor Ian Steward, who, seemed to deeply understand what the real problem about these models was:

> The equation itself wasn't the real problem. It was useful, it was precise, and its limitations were clearly stated. [...]
> The formula was fine if you used it sensibly and abandoned it when market conditions weren't appropriate. The trouble was its potential for abuse. It allowed derivatives to become commodities that could be traded in their own right. The financial sector called it the Midas Formula and saw it as a recipe for

> making everything turn to gold. But the markets forgot how the story of King Midas ended.
>
> -*Steward, I. "The mathematical equation that caused the banks to crash". The Guardian.* [7]

As there were not any other notorious incidents in the following years, and the main cause of the *Flash Crash* was still unclear, trading models did not become infamous and investment companies kept on betting on them, and in fact they still do it nowadays. But it was not until 2008 when a second big failure was caused by the same problem, even though then the detonator was different; The massive exchange of information led to the biggest amount of data ever held. That meant that if there was a slight change in the Stock Market, all the trading software would instantly know about it, and obviously would react according to the data recieved. The lack of human supervision and the abuse of automatition produced a derailment of all the positive trend these models supplied, being the Crash of 2008 the first recent proof, and *The Crash of 2:45*, in 2010, the second one. To keep in mind how damaging these crashes were, it is important to remark that the first one opened the doors to the 2007 Global Financial Crisis, and this last case led to the start of the current unpopular 2010 Eurozone Crisis.

After knowing about all these financial downfalls where modelling has been an important accomplice, one starts to wonder whether they should be used or not. The bottom line is that not using pricing models based on PDEs would be a waste of knowledge since they work perfectly under certain circumstances. So, should we blindly trust them? Absolutely never; The appropiate use for pricing models lies between a robust fitting and proper human suppervision, as it has been proved that a good use can end up handing in a really decent profit, being one of the most important tools for pricing and to predict risk.

Chapter 2

Modelling: Mean Field Theory

Focusing on analyzing the model proposed by the work of M.d.M.González, M.P.Gualdani and J.Solà-Morales [1], one wonders about the origin of the idea of modelling a big group of individuals who may interact with each other. This, as previously stated, comes from the Mean Field Gaming concept introduced, among others, by J.-M. Lasry and P.L. Lions [2].

The Mean Field Game Theory studies how to model a big group of individuals, from now on called population, who share a common interest and their decision might be determined by how others react. For example, the position of a big population of small particles that move around a three-dimensional space can be modeled through mean field theories since the particles hit each other and their position gets changed every time they collide.

2.1 A stable model: Diffusion

One well known method to model particles in space is the diffusion equation:

$$\frac{\partial \phi(r,t)}{\partial t} = \nabla[D(\phi,r)\nabla\phi(r,t)] \tag{2.1}$$

Where $\phi(r,t)$ is the density of the diffusing population at location r and time t and $D(\phi,r)$ is the collective diffusion coefficient for density ϕ at location r, but to make things more clear, we can assume the diffusion coefficient is a constant, so the system can be simplified as:

$$\frac{\partial \phi(r,t)}{\partial t} = D\frac{\partial^2 \phi(r,t)}{\partial x^2} \tag{2.2}$$

In this case we see how simple this model looks, modelling interaction with the second derivative term. Now, having a look at Lasry and Lions' work, one can see how they used mean field game theory to model a pricing predictive method:

> We introduce a simple mean-field model for the dynamical formation of a price. We consider an idealized population of players (which however somehow reflects the nature or microstructure of financial markets) consisting of two groups namely one group of buyers of a certain good and one group of vendors of the same good. Postulating some exogenous randomness in price preferences, we describe this population by two densities f_B, f_V i.e. nonnegative functions of (x,t) where t stands for time and x stands for a possible value of the price (roughly speaking $f_B(x,t)$ represents the number of potential buyers at a price x at time t). We denote by $p(t)$ the price resulting from a dynamical equilibrium and we assume that there is some friction measured by a positive parameter a (one could think of $2a$ to be the bid-ask spread). And we obtain the following system of mean-field equations:

$$\begin{cases} \frac{\partial f_B}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f_B}{\partial x^2} = \lambda\delta(x-p(t)+a), & \text{if } x < p(t), t > 0 \\ f_B \geq 0, f_B(x,t) = 0, & \text{if } x \geq p(t), t \geq 0 \end{cases} \tag{2.3}$$

$$\begin{cases} \frac{\partial f_V}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f_V}{\partial x^2} = \lambda\delta(x-p(t)-a), & \text{if } x > p(t), t > 0 \\ f_V \geq 0, f_V(x,t) = 0, & \text{if } x \leq p(t), t \geq 0 \end{cases} \tag{2.4}$$

$$\lambda = -\frac{\sigma^2}{2}\frac{\partial f_B}{\partial x}(p(t),t) = \frac{\sigma^2}{2}\frac{\partial f_V}{\partial x}(p(t),t) \tag{2.5}$$

> The multiplier λ measures the number of transactions at time t (i.e. the flux of buyers which must be equal to the flux of vendors). The parameter $\sigma > 0$ measures the randomness. And δ denotes either the usual delta function δ_0, or a smoothed version if it
>
> -*Lasry, J. M.; Lions, P. L. (2007). "Mean field games". Japanese Journal of Mathematics.* [2]

These equations give us an idea of how buyers and vendors interact with each other, and since there is a second derivative term, one understands that, as Lasry and Lions admitted, this is a diffusion model.

We can consider the randomness of the buyer and vendor density to follow a Gaussian random variable with variance σ^2, and a to be the transaction cost, so it's easy to understand that for a buyer, when $x < p(t)$, and more specifically, when $x < p(t) - a$ the chances of buying should be really high, and on the other hand, for a vendor when $x > p(t) + a$ the chances of selling should as well be really high. It is a matter of common sense to think that f_B and $-f_V$ work symmetrically as when one buys, automatically becomes a potential vendor, and reciprocally, when a vendor sells, automatically becomes a potential buyer.

We will assume that the delta terms take part in the equation representing the phenomena that when a buyer buys, it becomes a vendor at the point $x = p(t) + a$, and likewise when a vendor sells it becomes a potential buyer at $x = p(t) - a$.

This symmetry can be translated into a more simple model defining one new density unknown $f = f_B - f_V$ and thus:

$$f_t - \frac{\sigma^2}{2} f_{xx} = -\frac{\sigma^2}{2} f_x(p(t),t)(\delta(x + p(t) + a) - \delta(x - p(t) - a)) \tag{2.6}$$

One now wonders how this model behaves as time goes to infinity. We can analyze this by finding the stationary solutions solving the following equation:

$$f_{xx} = f_x(p(t),t)(\delta(x + p(t) + a) - \delta(x - p(t) - a)) \tag{2.7}$$

It is easy to see that the solution of this equation is of the form $f(x) = \gamma_\rho(x - p_0)$, as

$$\gamma_\rho = \begin{cases} -\rho x/a & |x| \leq a \\ \rho & x < -a \\ -\rho & x > a \end{cases} \tag{2.8}$$

According to this model markets always stabilize, which is in fact a direct contradiction to the market description given in Chapter 1.

Keeping in mind that Lasry and Lions' model works for a stable market, we want to find a way to modify it adding an instability to the model. This idea leads us to the following section.

2.2 An unstable model: Oscillations

Without adding a massive change to Lasry and Lions' model, P. Guidotti and S. Merino [12] propose adding a new term that would add instability to the stationary solution due to a Hopf bifurcation. Therefore, the said model now takes the folowing form:

$$\begin{cases} \frac{\partial f_B}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f_B}{\partial x^2} = (\lambda - Rp'(t))\delta(x - p(t) + a), & \text{if } x < p(t), t > 0 \\ f_B \geq 0, f_B(x,t) = 0, & \text{if } x \geq p(t), t \geq 0 \end{cases} \tag{2.9}$$

$$\begin{cases} \frac{\partial f_V}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 f_V}{\partial x^2} = (\lambda + Rp'(t))\delta(x - p(t) - a), & \text{if } x > p(t), t > 0 \\ f_V \geq 0, f_V(x,t) = 0, & \text{if } x \leq p(t), t \geq 0 \end{cases} \tag{2.10}$$

$$\lambda = -\frac{\sigma^2}{2}\frac{\partial f_B}{\partial x}(p(t),t) = \frac{\sigma^2}{2}\frac{\partial f_V}{\partial x}(p(t),t) \tag{2.11}$$

To simplify these systems, one can join them using, as previously seen, $f = f_B - f_v$ and then, the simplified system would be

$$f_t - \frac{\sigma^2}{2} f_x x = -\frac{\sigma^2}{2} f_x(p(t),t)(\delta_{p(t)-a} - \delta_{p(t)+a}) - Rp'(t)(\delta_{p(t)-a} + \delta_{p(t)+a}) \tag{2.12}$$

Where $\delta_a := \delta(x-a)$ is the Dirac delta function applied to $x = a$.

To find an expression for $p'(t)$ one has to first check that $f(p(t),t) = 0$, and it's clear since both f_B, f_V satisfy this property. Then if we evaluate the differential equation at $x = p(t)$, we get that $f_t(p(t),t) = \frac{\sigma^2}{2} f_{xx}(p(t),t)$, therefore:

$$\begin{aligned} 0 &= \frac{d}{dt} f(p(t),t) = f_x p'(t)(p(t),t) + f_t(p(t),t) \\ \Rightarrow f_x(p(t),t)p'(t) = -f_t(p(t),t) \Rightarrow f_x(p(t),t)p'(t) &= -\frac{\sigma^2}{2} f_{xx}(p(t),t) \end{aligned} \tag{2.13}$$

Thus one gets the following expression for $p'(t)$ that will be very handy for the numerical analysis of the system:

$$p'(t) = -\frac{\sigma^2}{2}\frac{f_{xx}(p(t),t)}{f_x(p(t),t)} \tag{2.14}$$

Considering that $p'(t)$ appears in the new set of equations, one can say that this model takes in consideration that the evolution of the prices affects the behaviour of the market, and this is why we can say that this new model is trend dependent.

2.2.1 R, or how to portray the market

The trend dependence in such a model makes a lot of sense considering that the first not-so naive idea that helped to formulate Lasry and Lions' model was that this concept suited the idea of the Mean Field Game Theory: a big group of individuals who interact with each other and may or may not modify the behaviour of the whole group.

To clarify that, let's think about a big flock of birds who fly around together and have to follow their immediate neighbours. They move in such a way that makes the flock move and create figures without letting any bird go away alone, because if one birds tries to be an outlier, his neighbour would follow him without breaking the flock. This phenomena is somehow related to our model, since the interaction of the individuals, even the slightest change, can modify the whole value of the market, and if someone buys or sells at a very high or low price, other people would follow him, or even if they don't, he would make the price tumble producing changes in the other buyers or sellers' decisions.

An instability was obtained adding the reaction term R to the system, but one now wonders about its meaning, since we are modelling a pricing phenomena, and every term should take part in a financial process. First of all, it's easy to see that if $R = 0$, we have the same model as Lasry and Lions, so in this case we already know how the market would behave. Let's now have a look at the model when R is either positive or negative:

The meaning of R > 0 is that if the prices grow ($p'(t) > 0$) then some buyers leave the market and at the same time some people outside the market (perhaps these previous buyers) enter as vendors. This is somehow the naive idea that when the prices are high it is time to sell, not to buy. And the contrary if $p'(t) < 0$.

It's clear that one could argue that there are other places, and not only $x = p(t) \pm a$, to leave or enter the market, but we think of this as the possibility that makes the simplest model.

But the case $R < 0$, being right the contrary, is also meaningful. If $p'(t) > 0$ (prices growing) it implies that some people outside the market enter into the game as buyers, perhaps because they feel that the prices may keep growing for some time, so it is a good moment to buy. And in the same situation ($R < 0$ and $p'(t) > 0$) some vendors leave the market, perhaps also expecting the prices to keep growing and re-enter into the market as vendors when the prices become higher. And the contrary, if $R < 0$ and $p'(t) < 0$.

In summary, somehow, $R > 0$ means conservative market, while a more aggressive investment is represented by $R < 0$. Without being precise, one can roughly say that our results with $R > 0$ will lead to oscillations and, on the contrary, $R < 0$ will lead to traveling waves, both inflationary and deflationary.

To locate the free boundary at $x = 0$, we can rearrange the system using a new variable $\omega(x,t) = f(p(t)+x,t)$, so without loss of generality, considering $\sigma = \sqrt{2}, a = 1$, our new system would look as the following equation:

$$\begin{cases} \omega_t = \omega_{xx} + p'(t)\omega_x - \omega_x(0,t)(\delta_{-1} - \delta_1) - Rp'(t)(\delta_{-1} + \delta_1) \\ \omega(0,t) = 0, \quad p'(t) = \frac{\omega_{xx}(0,t)}{\omega_x(0,t)} \end{cases} \tag{2.15}$$

M.d.M.González, M.P.Gualdani and J.Solà-Morales' paper suggests that when the equilibrium solutions become unstable sometimes happens that the function $\omega(x,t)$ loses the right sign near $x = -1$ and $x = 1$, becoming, respectively, negative and positive and then physically nonsense. To avoid this behaviour one can discuss the nonlinearity adding a function ϕ that must satisfy the following properties:

1. $\phi(1) = -\phi(-1) = 1$
2. $\phi(r) = -\phi(-r)$
3. $\phi(r) > 0$, when $r > 0$

Four possible candidates for ϕ could be $\phi(r) = sign(r)$, $\phi(r) = r$, $\phi(r) = \tanh(\rho r)/\tanh(\rho)$ or $\phi(r) = \tan^{-1}(\rho r)/\tan^{-1}(\rho)$. On the following chapters there will be a discussion about whether we should use one or another. So finally, our final system, and in fact the one that will be analysed later on is:

$$\begin{cases} \omega_t = \omega_{xx} + p'(t)\omega_x - \omega_x(0,t)(\delta_{-1} - \delta_1) - Rp'(t)(\phi(\omega(-1,t))\delta_{-1} - \phi(\omega(1,t))\delta_1) \\ \omega(0,t) = 0, \quad p'(t) = \frac{\omega_{xx}(0,t)}{\omega_x(0,t)} \end{cases} \tag{2.16}$$

2.2.2 A first look at the linear problem

To conclude, for a further analysis, one has to find the linear version of the problem. To do so, let's think about the stationary solution γ_ρ defined in 2.8, taking $\rho = 1$ so as to simplify.

Thus, being ω a solution of 2.16, one can write $\omega = \gamma_1 + \varepsilon g$. Substituting in 2.16, differentiating with respect to ε, and setting $\varepsilon = 0$, one gets

$$g_t = L_1 g \tag{2.17}$$

Where the operator L_1 is defined as

$$L_1 g = g_{xx} - g_{xx}(0,t)\chi_{(-1,1)} - g_x(0,t)[\delta_-1 - \delta_1] + Rg_{xx}(0,t)[\delta_-1 + \delta_1] \tag{2.18}$$

As long as ϕ satisfies the said properties, and $\chi_{(-1,1)}(x) = \gamma_x^1(x)/\gamma_x^1(0)$. Taking into consideration that $\omega(0) = 0$ one has to keep in mind that consequently for the perturbation part $g(0) = 0$.

Before studying the analytical properties of our model, one would like to know its behaviour and even try to figure out how and when interesting things happen. Having this interest opens new horizons to the project, leading to the next section, about numerical partial differential equations.

2.3 Slices of a PDE: Discretization

It's commonly known that when trying to solve a PDE numerically, one has to use a discretization method. There are some very popular and different techniques to do so, but we found the finite differences method the most appropiate way to study numerically the given model. One can say that given a problem with an unbounded domain, for example $t \in [0, \infty),\ x \in (-\infty, \infty)$, the finite differences method can be quite unfortunate since for a bounded and finite number of points of discretization, one would need a pair of auxiliary boundary points with boundary values, and one can not archieve this if the conditions are unbounded.

To deal with this situation, we have to think about our model and discuss whether it is appropiate or not to approximate the boundary conditions with a pair of auxiliary boundary values. If we first think about the physical phenomena that is being modeled, we get the idea that for values of $x \in (-a,a)$, the chances of making a good decision are quite unclear, so thinking that the probability for a buyer or a seller to sell or buy are definitely not black or white; but what happens if $|x| > a$?

If one is a potential seller and $x > a$, it leads to the situation of an appearance of a potential buyer who would actually buy your asset for a price $p > p(t) + a$ which means that you encounter what is commonly known as a *free meal*: the oportunity of selling an asset for an amount higher than what the market stablishes. Then one would imagine that the chances for a potential seller to sell under this situation are massive, thus the solution of the model would be really close to 1 for values of $x > a$, and $x < -a$ respectively for sellers. So, even though the boundary conditions are $\omega(-\infty) = -\omega(\infty) = 1$, we can consider two outlier values $-b, b$ such that $-b < -a < a < b$ as the new auxiliary boundary conditions without losing the physical meaning that will make the numerical analysis using the finite difference much more handy.

Once the boundary conditions are clear for the numerical version of the model, one should wonder how the newly discretized system would look like for a mesh of $n+1$ equispaced points in the interval $(-b,b)$. Thus, for the approximate values of ω, one gets the following expression:

$$\begin{cases} \omega_t^i = \omega_{xx}^i + p'(t)\omega_x^i - \omega_x(0,t)(\delta_{-1} - \delta_1) - Rp'(t)(\phi(\omega(-1,t))\delta_{-1} - \phi(\omega(1,t))\delta_1) \\ \omega(0,t) = 0, \quad p'(t) = \frac{\omega_{xx}(0,t)}{\omega_x(0,t)} \\ \omega^i := \omega(t,x_i),\ i \in \{0,..,n\}, \quad x_0 = -b, \quad x_n = b, \quad x_{n/2+1} = 0 \end{cases} \tag{2.19}$$

Now that the system is discretized and sliced into a system of $n+1$ ODEs with defined finite boundary conditions, one can aaply a numerical method to find a solution. To do so we introduce the matrix form of the system:

$$\vec{\omega}'(t) = A_1\vec{\omega} + b_1 + p'(t)(A_2\vec{\omega} + b_2) - \omega_x(0,t)\vec{c} - Rp'(t)\vec{d} \tag{2.20}$$

To calculate A_1, A_2, b_1, and b_2 we have to think about the Taylor's expansion for the first and second derivative, giving us a first and second order approximation:

$$\omega_x^i \approx \frac{\omega^{i+1} - \omega^{i-1}}{2h}, \quad \omega_{xx}^i \approx \frac{\omega^{i+1} + \omega^{i-1} - 2\omega^i}{h^2} \tag{2.21}$$

And as well for the case $\omega_x(0,t)$:

$$\omega_x(0,t) \approx \frac{\omega(x_{n/2+2},t) - \omega(x_{n/2},t)}{2h} \tag{2.22}$$

Considering $h = \frac{2b}{n}$ as the space between two consecutive points of the mesh.

Thus,

$$A_1 = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}, \quad b_1 = \frac{1}{h^2} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix} \tag{2.23}$$

$$A_2 = \frac{1}{2h} \begin{bmatrix} 0 & 1 & & \\ -1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{bmatrix}, \quad b_2 = -\frac{1}{2h} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.24}$$

And considering that, since the area of the δ function has to be equal to 1, $\delta = 1/h$ if and only if $x = 1$ and zero otherwise, we can as well compute the vectorial form of the function as:

$$c = \begin{bmatrix} 0 \\ \vdots \\ -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ \vdots \\ \phi(\omega(-1,t)) \\ 0 \\ -\phi(\omega(1,t)) \\ \vdots \\ 0 \end{bmatrix} \tag{2.25}$$

Giving these expressions one could solve the system numerically using MatLab and its function `ode45`. To do so we would need to first specify the values of R, a and b. From now on, $a = 1$, $b = 2$, even though different values could be chosen and the final ideas would be the same; In this specific case we will choose $R = 8$. To go on, one should define an initial condition. This would have to be a decreasing function $\omega(x,0)$ such that $\omega(0,0) = 0$, and $\omega(-2,0) = -\omega(2,0) = 1$.

As a first glimpse, chosing $\omega(x,0) = -\tanh(5x)/\tanh(10)$, integrating the system in a mesh of 17 points for time $t \in [0,100]$ one gets the following result as a solution for the system at time $t = 100$, which as desired resembles the expected stationary solution γ^1 defined in 2.8.

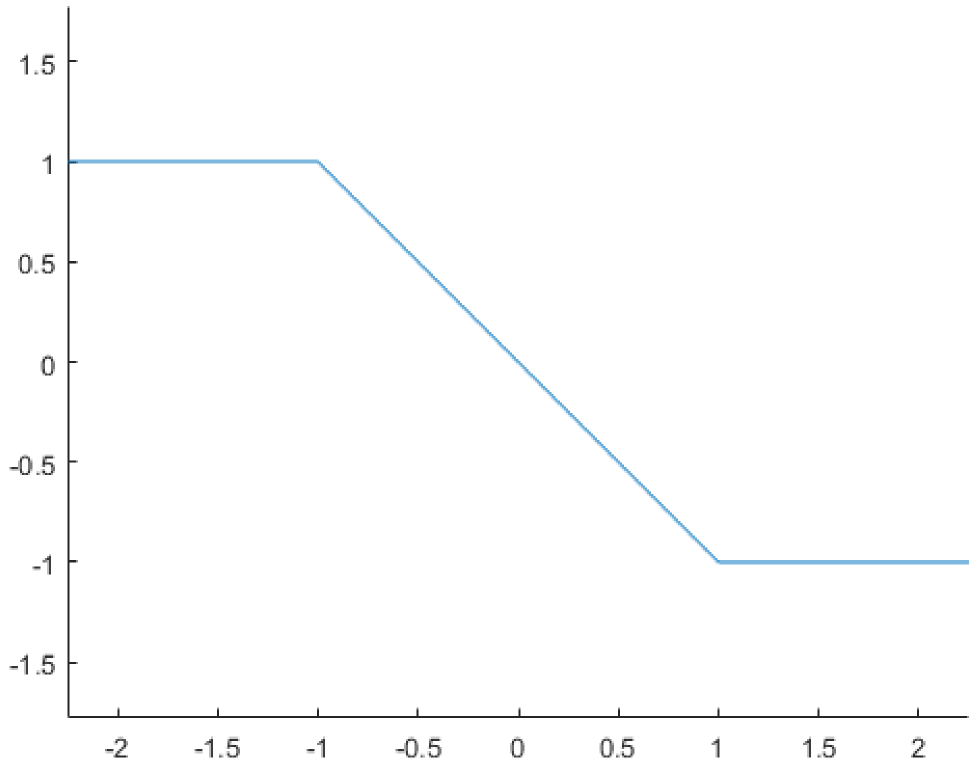

Fig. 2.1 Plotted solution for t=100

Chapter 3

Analysis

Before starting the chapter, let us recapitulate the main results stated in the last chapter.

3.1 A rundown

To briefly summarize the last chapter, one has to keep in mind that when defining the model, the equations lead to a partial differential equation. This equation could be discretized as a system of n ordinary differential equations that simplified the process of finding numeric solutions. This new system looked as follows:

$$\begin{cases} \omega_t^i = \omega_{xx}^i + p'(t)\omega_x^i - \omega_x(0,t)(\delta_{-1} - \delta_1) - Rp'(t)(\phi(\omega(-1,t))\delta_{-1} - \phi(\omega(1,t))\delta_1) \\ \omega(0,t) = 0, \quad p'(t) = \frac{\omega_{xx}(0,t)}{\omega_x(0,t)} \\ \omega^i := \omega(t,x_i),\ i \in \{0,..,n\}, \quad x_0 = -b, \quad x_n = b, \quad x_{n/2+1} = 0 \end{cases} \tag{3.1}$$

Having a system of ODEs simplifies both the analytic and numeric results being easier to find conclusions. First of all, and given that one can find results of the system of ordinary differential equations using mathematical software, we saw that for some conditions including $R = 8$, one gets convergence for large t to the stationary solution as shown in 2.1. But then one wonders what happens if we try to solve it for different values of R, for example $R = 20$. The following figure shows how the solution becomes unstable as the results for time $t = 101, \ldots, 110$ seem to oscillate as time increases differing from the stationary solution we got for $R = 8$ and $t \geq 100$.

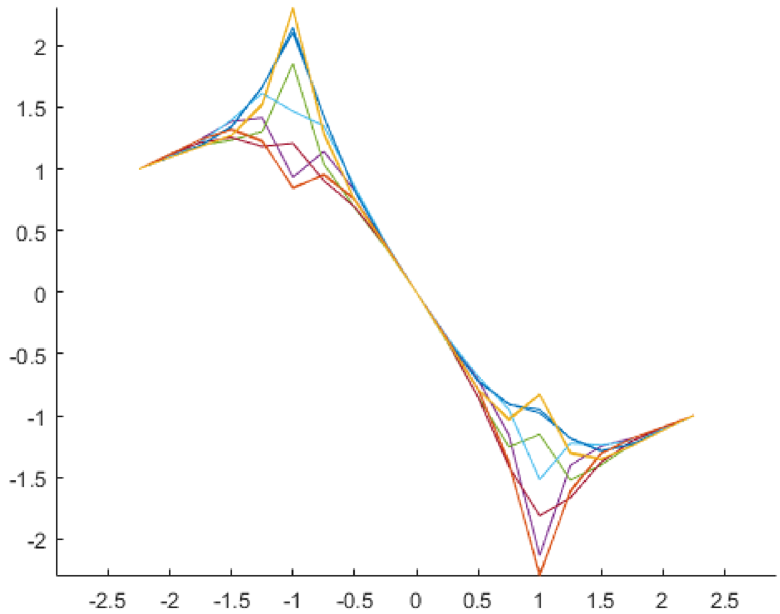

Fig. 3.1 Plotted solution for $t \in [101, 110]$ and $R = 20$

These oscillations are the instabilities wanted in M.d.M.González, M.P.Gualdani and J.Solà-Morales' system defined in 2.16. To study this behaviour one should look back at the linearised system 2.17 and still taking into account that the operator L_1 looks as follows:

$$L_1 g = g_{xx} - g_{xx}(0,t)\chi_{(-1,1)} - g_x(0,t)[\delta_-1 - \delta_1] + Rg_{xx}(0,t)[\delta_-1 + \delta_1] \qquad (3.2)$$

One of the bottom lines of M.d.M.González, M.P.Gualdani and J.Solà-Morales' paper is the next theorem that will help us understand what happens when varying the R value in the system of differential equations.

Theorem 3.1.1 *The eigenvalues λ of the linear part of 3.1 around $\omega = \gamma^1$, given by*

$$L_1 g = g_{xx} - g_{xx}(0,t)\chi_{(-1,1)} - g_x(0,t)[\delta_{-1} - \delta_1] + Rg_{xx}(0,t)[\delta_{-1} + \delta_1]$$

satisfy

$$\Re(\lambda) \leq 0 \text{ for all } -1 \leq R \leq R_0,$$

where $R_0 \simeq 9.36... = (1 - e^{a_0}\cos(a_0))/a_0$ and a_0 is the root of $\cos(a) - \sin(a) = e^{-a}$ near $a \simeq 3.94...$.

For $R = -1$ a real eigenvalue becomes positive, and for $R = R_0$ a pair of simple complex conjugated nonzero eigenvalues cross the imaginary axis from left to right.

Claim 3.1.1 *Roughly speaking, for $R = -1$ the bifurcated solutions are traveling waves, moving both right and left, and for $R = R_0$ an Andronov-Hopf type bifurcation occurs, giving rise to periodic oscillations.*

Claim 3.1.2 *At $R = R_0$ a family of periodic solutions does appear near γ^1. These numerical simulations show at least that for $\phi = \phi_3$ the bifurcation is supercritical, and stable oscillations seem to persist for all $R > R_0$.*

3.2 An introduction to stability and bifurcation

Proving Theorem 3.1.1 is one of the main goals of M.d.M.González, M.P.Gualdani and J.Solà-Morales' paper, and it is let as a recommended lecture as to get deeper in this topic and to understand some properties of the solution of the system.

The theorem itself defines a region $-1 \leq R \leq R_0$ such that all the eigenvalues of the linearised system around the stationary solution are non-positive. As Claim 3.1.1 states, this might be a good evidence of a Hopf Bifurcation for values of $R \simeq 9.36$.

As to make things easier for the reader, some concepts will be introduced, since these definitions are very important to understand and develop all the mathematical process involved in this dissertation.

3.2.1 Stability

In order to understand the following discussion about whether a price should or should not stabilise one has to understand the physical meaning of stabilisation, which has been discussed in Chapter 2. Moreover, to discuss such a thing analytically, a definition of stability has to be set. Since the model has ended up being a system of n ordinary differential equations, i.e. a differential equation in $\mathbb{R}^n$ we can summarise the most important results about stabilisation in multidimensional spaces so to start a deeper analysis about the model's properties.

To make things clear, there are a several different definitions involving stability, but for this matter of studies the most appropiate defnition is the one about Lyapunov Stability.

Definition 3.2.0.1 *Consider the differential equation*

$$\dot{x} = f(x,t), \quad x \in R^n \tag{3.3}$$

A point x is Lyapunov stable iff for all $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $|x - y| < \delta$ *then*

$$|\phi(x,t) - \phi(y,t)| < \varepsilon$$

for all $t \geq 0$

Definition 3.2.0.2 *A point x is quasi-asymptotically stable iff there exists* $\delta > 0$ *such that if* $|x - y| < \delta$ *then* $|\phi(x,t) - \phi(y,t)| \to 0$ *as* $t \to \infty$

Definition 3.2.0.3 *A point x is asymptotically stable iff it is both Liapunov stable and quasi-asymptotically stable.*

These first definitions will not be used explicitly, but it historically led to several stability results such as the following, regarding linear stability for linearised systems:

Theorem 3.2.1 *Suppose that the non-linear equation* $\dot{x} = f(x)$ *has* $f(0) = 0$ *and a linearisation* $\dot{x} = Ax$ *at* $x = 0$*. If A has n distinct eigenvalues, each of which has strictly negative real part, then* $x = 0$ *is asymptotically stable for the linearized system* $\dot{x} = Ax$*. And in fact this last result remains true even if the eigenvalues of A are not distinct; it is sufficient that A has eigenvalues with strictly negative part, then* $x = 0$ *is asymptotically stable.*

These results have a very interesting meaning but can hardly adjust to the system we are working with since they are refering to linear systems. The thing is that results regarding nonlinear systems do not difer much from the linear theory, and this introduction is actually very valuable and a good place to start talking about our system's stability.

From now on the idea is to prove how the linear results are still valid for the nonlinear case (at least locally) and to show how stability and manifolds' properties preserve under the nonlinear case.

Starting under the assumption that x_0 is a stationary point of the equation $\dot{x} = f(x), \quad x \in R^n$ one can think about a shift in the coordinate system so as to set the stationary point to be at the origin, thus $f(0) = 0$. Assuming f is smooth, we can expand it about the origin as a Taylor's expansion as

$$\dot{x} = Df(0)x + O(|x|^2) \tag{3.4}$$

And simplifying the expression for the terms of order $|x|^2$ and higher, we get the following linear differential equation.

$$\dot{x} = Df(0)x \tag{3.5}$$

We sure are one step closer to mixing the linear results with the nonlinear scenario, but a local change of coordinates that brings the nonlinear equation to the linear equation is a very optimistic thing to ask for and turns the problem into a complicated issue.

Since the algebraic work differs a bit from this work's goal, Poincaré's linearisation theory is left as an interesting reading since it helps the reader understand how the linear and nonlinear results can be linked.

Definition 3.2.1.1 *A stationary point x_0 is said to be hyperbolic iff Df(x) has no zero or purely imaginary eigenvalues.*

There exists an extensive theory of hyperbolic stationary points, but for our case we will state two very important results:

Theorem 3.2.2 *(STABLE MANIFOLD THEOREM) [3]*
Suppose that the origin is a hyperbolic stationary point for $\dot{x} = f(x)$ and E^s and E^u are the stable and unstable subspaces of the linear system $\dot{x} = Df(0)x$. Then there exist local stable and unstable manifolds $W^s_{loc}(0)$ and $W^u_{loc}(0)$ of the same dimensions as E^s and E^u respectively.

Theorem 3.2.3 [3] *Suppose x_0 is a stationary point of the equation $\dot{x} = f(x)$, f is smoth and all the eigenvalues of $Df(x_0)$ are strictly negative. Then x_0 is asymptotically stable.*

The proofs of both theorems are quite extensive and are detailed in [3], but putting them forward will be very handy when the Hopf Bifurcation is introduced. Even though the proofs are hard, one can easily see how Theorem 3.2.2 guarantees that locally, theorems for linearized systems work as well for the non-linear case. Thus, applying Theorem 3.2.3, for every stationary point of the system 2.16, in particular $x = 0$, we can say that at least locally, this said point is asymptotically stable, which is an important result that will lead to the discussion of the existance of a bifurcation.

3.2.2 Bifurcation

Bifurcation theory is the turning point of our problem. It is what is behind the change of the system's behaviour, from stability to instability, and makes us wonder if the model is appropiate, valid and under what circumstances. It is clear that a precise pricing model should never stabilize, so the stable solutions obtained for certain values of the R value in our model should be automatically discarded. The question is: can we make a distinction between valid and invalid results without having to solve the problem? The answer is yes, and is led to the analysis of the problem's bifurcation. Once we are able to determine its properties and define it, the model ill be much more useful and meaningful. This is why we should first introduce the concept of bifurcation.

Bifurcation theory describes the way that topological features of a flow, such as the number of stationary points and periodic orbits, vary as one or more parameters are varied.

The fundamental observation for stationary points of flows is that if the stationary point is hyperbolic, i.e. the eigenvalues of the linearized flow at the stationary point all have non-zero real parts, then the local behaviour of the flow is completely determined by the linearized flow. Hence bifurcations of stationary points can only occur at parameter values for which a stationary point is non-hyperbolic.

This is a great way to create mechanisms to find bifurcations numerically, and in fact the mechanism for finding the critical values for parameters in a bifurcation in Chapter 4 arises from this last paragraph.

The main tool to study such bifurcations is the non-hyperbolic equivalent of the Stable Manifold Theorem [3.2.2] called the Centre Manifold Theorem. This generalizes the idea of the centre manifold for linear systems to nonlinear systems and can really helpful when studying bifurcation problems.

Theorem 3.2.4 *(CENTRE MANIFOLD THEOREM) [3]*
Let $f \in C^r(R^n)$ with $f(0) = 0$. Divide the eigenvalues, λ, of $Df(0)$ into three sets, σ_u, σ_s, and σ_c, where $\lambda \in \sigma_u$ if $Re(\lambda) > 0$, $\lambda \in \sigma_s$ if $Re(\lambda) < 0$, and $\lambda \in \sigma_c$ if $Re(\lambda) = 0$.
Let E^u, E^s and E^c be the corresponding generalized eigenspaces. Then there exist C^r unstable and stable manifolds (W^u and W^s) tangential to E^u and E^s respectively at $x = 0$ and a C^{r-1} centre manifold, W^c, tangential to E^c at $x = 0$. All are invariant, but W^c is not necessarily unique.

3.3 The Hopf Bifurcation

A Hopf bifurcation is a critical point where a system's stability switches and a periodic solution arises. More accurately, it is a local bifurcation in which a fixed point of a dynamical system loses stability, as a pair of complex conjugate eigenvalues of the linearization around the fixed point cross the complex plane imaginary axis.

The Hopf Bifurcation is not a simple case of a bifurcation, in fact harder to analyse since it involves a non-hyperbolic stationary point with linearized eigenvalues $\pm i\omega$ and thus, a two-dimensional centre manifold plus the bifurcationg solutions are periodic rather than stationary. This usually leads to some serious algebraic operations to manage to define its properties and characteristics.

In order to classify the two types of Hopf bifurcations one should recap the instability theory stated before, as the two instabilities arising from the bifurcation show different behaviours:

A Hopf Bifurcation is said to be supercritical if as a constant R is increased, a sink changes to a source expelling a limit cycle. The said bifurcation is called subcritical if the limit cycle is in this case absorbed.

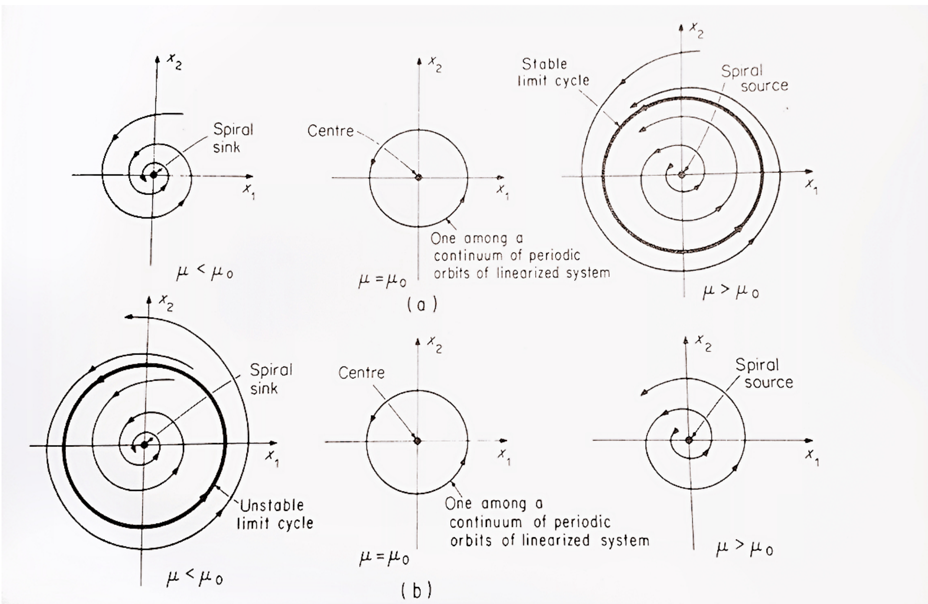

Fig. 3.2 a) Supercritical bifurcation b) Subcritical bifurcation

The main result to define whether a bifurcation is in fact a Hopf Bifurcation or not is the Hopf Bifurcation Theorem. Using this important result will be crucial when evaluating our model in Chapter 4 because will define conditions for the constant value to determine when instabilities occur.

Theorem 3.3.1 *(HOPF BIFURCATION THEOREM) [11]*
Let f^{μ} be a vector field on R^n, $n \geq 2$, parametrized by $\mu \in R$ and C^k, $k \geq 4$ jointly in $x \in R^n$ and μ. Suppose $f^{\mu}(\hat{x}(\mu)) = 0$ for a locally unique point $\hat{x}(\mu)$ and write J^{μ} for the Jacobian $(Df)_{\hat{x}(\mu)}$. Suppose

- *J^{μ} has a pair of complex conjugate eigenvalues $\lambda(\mu)$, $\overline{\lambda}(\mu)$ for which $Re\lambda(\mu) = 0$ at $\mu = \mu_0$ and*
$$\frac{d}{d\mu} Re\lambda(\mu) > 0,\ Im\lambda(\mu) > 0$$
at $\mu = \mu_0$;
- *Every eigenvalue $\nu(\mu)$ of J^{μ} except $\lambda(\mu)$ and $\overline{\lambda}(\mu)$ satisfies*
$$Re\ \nu(\mu_0) \neq 0;$$
- *Re ψ, found from 3.6 below, is nonzero.*

Then there is a range either of positive or of negative values of $\Delta\mu \equiv \mu - \mu_0$, in which every value of μ corresponds to a unique limit cycle at a distance $O(|\Delta\mu|^{1/2})$ from $\overline{x}(\mu)$, and of period $2\pi/Im\ \lambda(\mu_0) + O(\Delta\mu)$. Furthermore,

- *If Re $\psi < 0$ and Re $\nu(\mu_0) < 0\ \forall \nu$, the limit cycle is attracting, while if Re $\psi > 0$ and Re $\nu(\mu_0) > 0\ \forall \nu$, the limit cycle is repelling.*

The curvature coefficient is *Re* ψ where

$$\psi = u_p v_j v_k \overline{v}_l f^p_{jkl} - 2 f^p_{jm} J^{-1}_{mq} f^q_{kl} - f^p_{lm}(J - 2i\omega)^{-1}_{mq} f^q_{jk} \tag{3.6}$$

Here $J = J^{\mu_0}$ and u^T and v are respectively left and right eigenvectors of J belonging to $\lambda(\mu_0)$ normalized so that $u^T v = 1$. Repeated subscripts imply sumation from 1 to n and f^p_{jk} means $\partial f^{\mu}_p(x)/\partial x_k \partial x_j$ evaluated at $x = \hat{x}(\mu_0)$, $\mu = \mu_0$.

The proof of these results is a very difficult yet interesting excercise that involves serious algebra. To avoid diverging form the main goal of this paper, let's see the results stated in a simple and practical example.

3.3.1 The classical Van der Pol oscillator

In dynamics, the Van der Pol oscillator is a non-conservative oscillator with non-linear damping. It evolves in time according to the second-order differential equation:

$$\frac{d^2y}{dt^2} - (\mu - y^2)\frac{dy}{dt} + y = 0 \tag{3.7}$$

where y is the position coordinate which is a function of the time t, and μ is a scalar parameter indicating the nonlinearity and the strength of the damping.

Let us consider the equations for the Van der Pol oscillator written as a system of ordinary differential equations:

$$\begin{cases} \frac{dx}{dt} = -y + (\mu - y^2)x, \\ \frac{dy}{dt} = x \end{cases} \tag{3.8}$$

It's easy to see that $(x,y) = (0,0)$ is an equilibrium point. Thus, the Jacobian matrix at the equilibrium associated to this system follows:

$$J = \begin{pmatrix} \mu & -1 \\ 1 & 0 \end{pmatrix} \tag{3.9}$$

It is easy to see that the characteristic polynomial of the linearization is equal to:

$$P(\lambda) = \lambda^2 - \mu\lambda + 1 \tag{3.10}$$

and so, the eigenvalues depending on μ are

$$\lambda_{1,2}(\mu) = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

And it's clear that $|\mu| \geq 2 \Rightarrow \lambda \in \mathbb{R}$ but for $|\mu| < 2$, $Im\ \lambda \neq 0$. Now one should wonder for which value of μ, $Re\ \lambda = 0$:

$$Re\ \lambda = 0 \Leftrightarrow \mu/2 = 0 \Leftrightarrow \mu = 0 \tag{3.11}$$

To check the last condition of the theorem we have to calculate the derivative of the real part of the eigenvalue:

$$\frac{d}{d\mu} Re\ \lambda = 1/2 > 0 \tag{3.12}$$

So one can finally say that for $\mu = 0$ a Hopf Bifurcation takes place since all the conditions of theorem 3.3.1 are satisfied. By the definition of instability one can see how the stability of the system changes and by 3.2.4 one can say that now there exist two unstable manifolds and $n-2$ stable manifolds. Now we have to check the behaviour and properties of the bifurcation, to do so let's apply theorem 3.3.1. Since all the conditions are satisfied for $\mu = 0$, we just have to determine wether the bifurcation is supercritical or subcritical using ψ as in 3.6.

The jacobian matrix of the linearization applied to $\mu = \mu_0 = 0$ is

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with eigenvalues $\lambda = \pm i$ and normalised eigenvectors as follow

$$v = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

Thus, applying the idea of 3.6 in system 3.8 applied at $\mu = 0$, calculating ψ numerically as in Appendix B.4, one gets $\psi = -2$, and as $Re\ \psi < 0$ one can say that the Hopf Bifurcation in 3.8 is supercritical.

Chapter 4

Numerical Analysis

The main goal of this paper is to prove 3.1.1 using numerical methods and analysis. To do so, one has to begin modelling the system and finding an appropiate numerical method to integrate it. As stated in Section 2, discretizing the system into $n+1$ ordinary differential equations makes the numerical analysis much easier. Thus, the following equation was found

$$\vec{\omega}'(t) = A_1\vec{\omega} + b_1 + p'(t)(A_2\vec{\omega} + b_2) - \omega_x(0,t)\vec{c} - Rp'(t)\vec{d} \tag{4.1}$$

And as previously explained, using MatLab and the function `fsolve`, one gets a possible solution depending on t for a fixed constant R, a fixed function ϕ and a fixed initial condition.

Now one should wonder about the Hopf Bifurcation studied in Chapter 3 and how this R parameter infuences the characterization of the stability of the system.

From now on, without loss of generality and unless stated, the function $-\tanh(5x)/\tanh(10x)$ will be picked as the initial condition, and $\phi(w) = \tanh(w)/\tanh(1)$ will be the chosen non-linearity to study the system.

4.1 Having a look at the results

As we already have the intuition that for values of $R \simeq 9.36$ there can be a possible Hopf Bifurcation, let's compute several results for $t = 100$ and $2n+1 = 17$.

In Figure 4.1 one encounters an unexpected fact since the analytical results suggested that the Hopf Bifurcation occurred near $R = 9.36$. This makes us wonder if the mesh is not

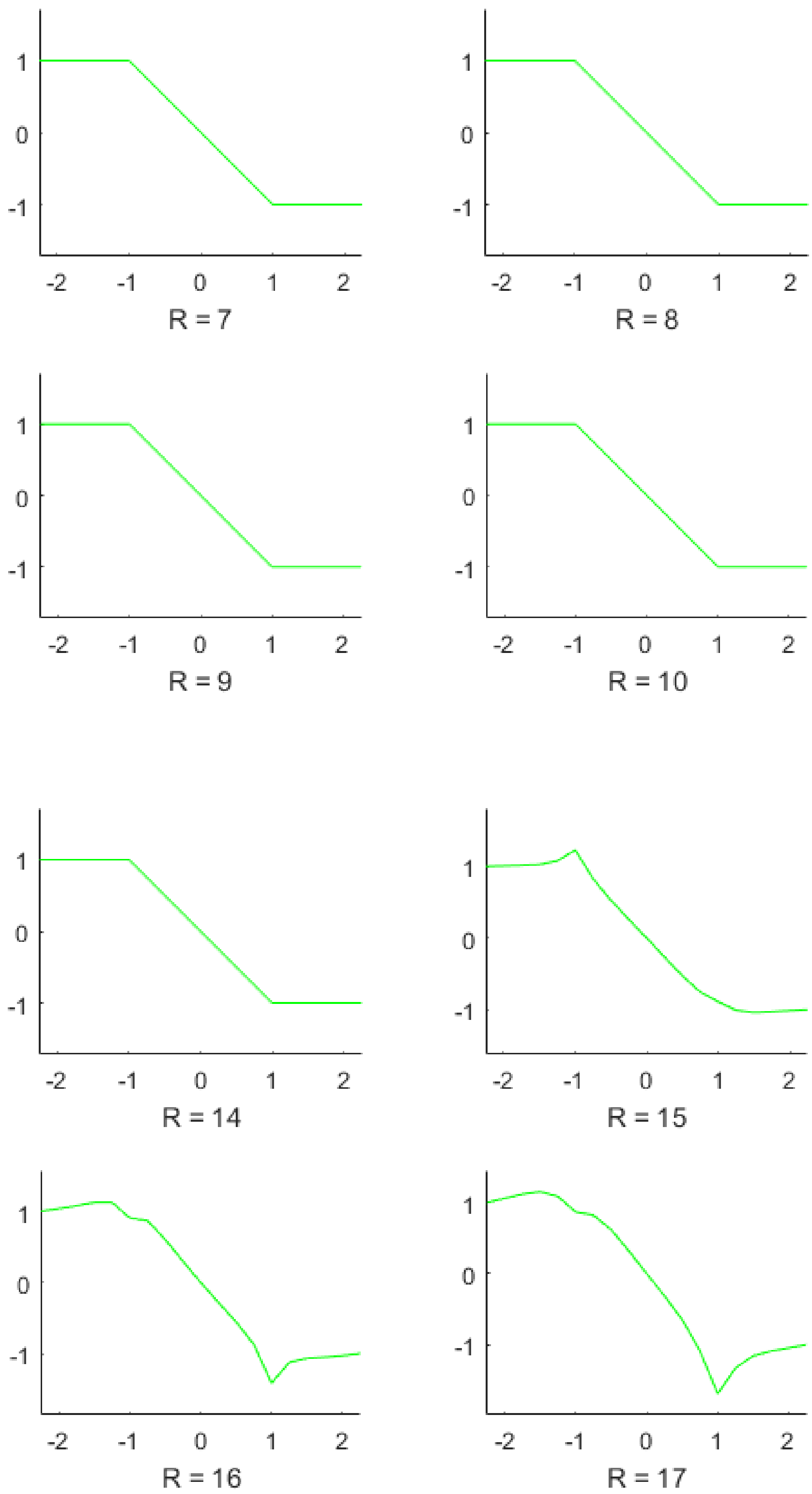

Fig. 4.1 Plotted solution for $R \in [7, 17]$ using a mesh of 17 nodes

accurate enough, and this is why we have to recompute the results for a bigger number of points, for example $2n+1 = 161$

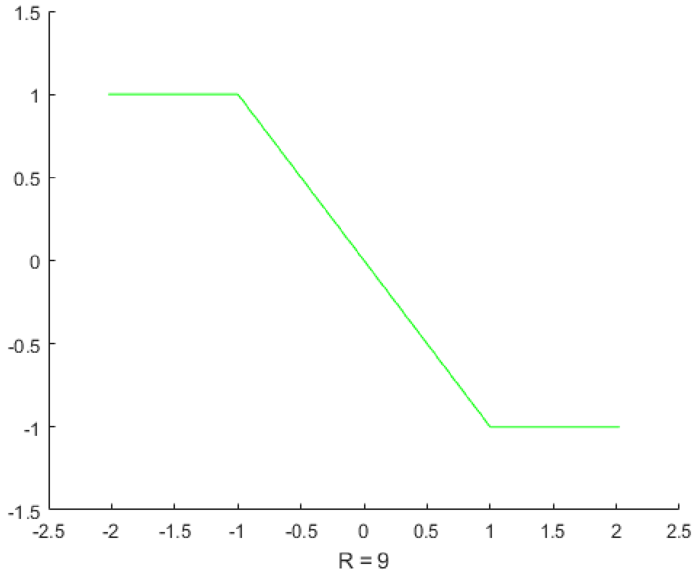

Fig. 4.2 Plotted solution for $R = 9$ and $t = 100$ using a mesh of 161 nodes

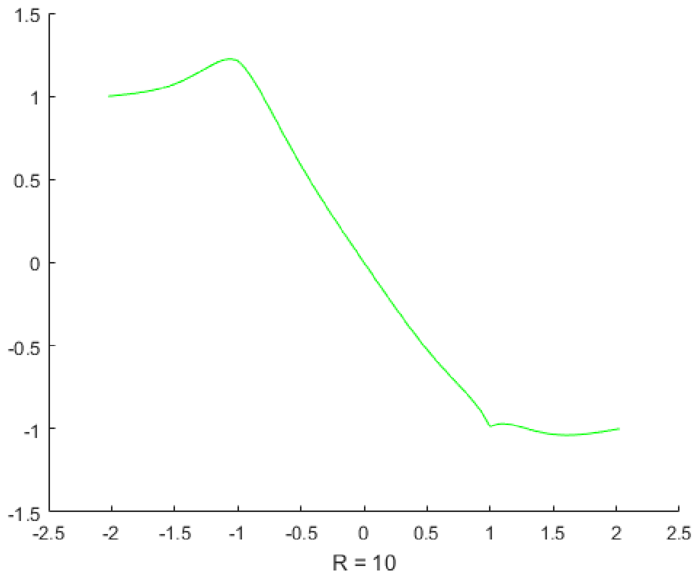

Fig. 4.3 Plotted solution for $R = 10$ and $t = 100$ using a mesh of 161 nodes

In Figures 4.2 and 4.3 one sees how the result seems more appropiate since it satisfies the expected idea that $9 < R < 10$. This is the first time that we have to take into account how important the accuracy of the mesh is. Hence, if the mesh is not good enough, one can obtain wrong results that have no meaning and can not be taken as valid information to make reasonable conclusions about the problem.

Now, considering that a mesh of size 161 is good enough, we should check that the critical value R is indeed 9.36. To do so we try two different methods:

1. Integrating the system using a good mesh even though the computational time could be very large.

2. Following the analytical results, try to find the Jacobian matrix and find the point R where for the first time a pair of eigenvalues will cross the imaginary axis from $Re(x) > 0$ to $Re(x) < 0$. This idea comes from the fact that a solution is stable if all the real parts of the eigenvalues of the Jacobian matrix of the linearised problem are strictly negative [3.2.4].

4.1.1 Brute force

Brute force, this is the idea of the first method. Finding a good mesh to see that for values of R around 9.36 the Hopf Bifurcation takes place. This is a pretty simple method but the computational time can easily become huge.

Our results were a bit decieving. For a mesh consisting of 401 points, the result was not as accurate as desired. Even though the obtained critical value did not diverge much from de desired one, this option made the process of finding R really slow. This made us think that looking for a better mesh would be the clue to this problem, but as the computational increased heavily we decided to give up the brute force method with a decent but not perfect result. This led to the second method.

4.1.2 Applying the analytical results

The idea of this section is to mix up the theoretical results, the definition of stability, and the idea of the Hopf Bifurcation to actually find the critical point.

To start, let's picture in Figures 4.4 - 4.6 the eigenvalues of the Jacobian matrix of the linearised problem using a mesh of 41 points.

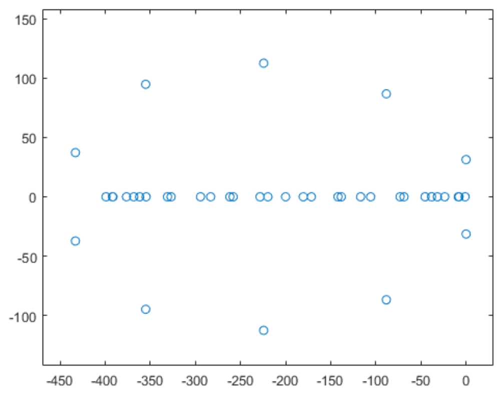

Fig. 4.4 Eigenvalues of the jacobian matrix for $R = 10$

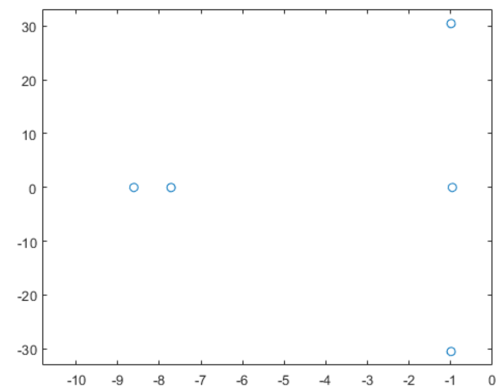

Fig. 4.5 Closest eigenvalues to the imaginary axis for $R = 9$

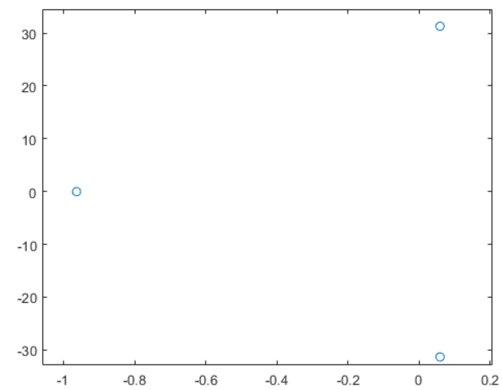

Fig. 4.6 Closest eigenvalues to the imaginary axis for $R = 10$

It's easy to see that the real parts of all the eigenvalues when $R = 9$ are strictly negative, and for $R = 10$ there is a pair of eigenvalues whose real part becomes positive. This fact satisfies the idea of instability defined in Chapter 3 and the conditions and results of the Centre Manifold Theorem [3.2.4], so creating an algorithm based on the bisection method would bound the critical point.

The code would basically look like this:

```
start = 1; final = 50;
niter = 50;
for i = 1:niter;
    m = (start+final)/2;
    res = maxEig(m);
    f0 = maxEig(start);
    ff = maxEig(final);
    if (sign(f0) == sign(res))
        start = m;
    else
        final = m;
    end
end
R_Critica = m
```

As `maxEig` returns the real part of the eigenvalue with the highest real part.

With this method and a mesh of $2n+1 = 401$ points we find $R = 9.3666$ as desired. So we can say that there is evidence of a Hopf Bifurcation appearing around $R = 9.36$. This method presents us a more appropiate result using a lighter mesh in a shorter amount of time but even though the computational time was much shorter than the first method's, it was definitely not very brief. This is because the program generates `niter` times the eigenvalues of a $2n+1$ dimensional matrix, which is not a cheap thing to do.

As a first result, by now, we can say that there is evidence of a possible Hopf Bifurcation at $R \approx 9.36$, but theoretically one can not admit this result as a true fact since all the evidence comes from numerical results. In order to fix this uncertainty we will mix the analytical results with the numerical experience in the following section.

4.2 Finding the Hopf characterization

Even though there is evidence of a Hopf Bifurcation occurring at $R = R_0 \approx 9.36$, given the complexity of the problem it is difficult to admit with total security that the bifurcation taking place is a Hopf Bifurcation. As recently announced, mixing the theoretical results with the numerical data would bound the conjecture proving the main Claim of this paper 3.1.1.

Right now, the main goal to conclude all the analysis is to prove that the bifurcation is in fact of a Hopf type, which is exactly what the claim in Chapter 3 3.1.1 stated. Once we proved that the bifurcation is actually a Hopf Bifurcation, there will be a discussion about its properties and its type, and whether it is a subcritical or a supercritical bifurcation.

In order to determine so, one has to look at the Hopf Bifurcation Theorem 3.3.1. Looking at figures 4.5 and 4.6 and checking the numerical results, which in fact is just analytical algebra computed numerically, we see how every condition of the Theorem is satisfied. This gives us the right to declare that we are now able to check the last condition of the theorem 3.6.

Since the formula to define ψ is actually composed by six sums from 1 to n, the computational time is of order $O(n^6)$, and going a bit back, let's remember that one of the first results was that the mesh to determine an accurate enough result had to be of order $n = 41$. This makes us guess that the complexity of this problem is indeed very high since the number of operations that take place is around $O(41^6) = O(10^9)$.

Taking into account that some of the operations that take place include symbolic differentiation and inverting matrices, to reduce that computtaional time severely we introduced a dynamic programming method to save operations keeping the results inside a 3-dimensional matrix of size n. This simplifies the cost and lets us try our problem for more competitive meshes bounding the results nicely.

The simplified pseudocode applying the said dynamic programming is as follows:

```
% We first get the position of the critical eigenvalues in
    the matrix of eigenvectors and keep them.
% Then we normalise the selected eigenvectors. Here we check
    that u*v=1.
% We set the matrices to apply the dynamic programming.
```

```
% Six consecutive sums keeping the values in the auxiliar
    matrices and updating the value for psi.
```

The actual code can be found in the file `eigen_jacobiana.m` and before running it, the initial conditions for the problem must be set in the file `disc1DR5.m`.

To conclude, we tried to get the value of ψ using a mesh of size 21. The computational time was still massive but we were able to get a result: Using $\phi(w) = \tanh(w)/\tanh(1)$ as a non-linearity, we got that $\psi = -27.5709 + 87.2966i$, with $Re\ \psi < 0$ suggesting that the bifurcation is a Supercritical Hopf Bifurcation. On the other hand, using $\phi(w) = w$ as a non-linearity, the result was $\psi = 1.3400 - 23.5090i$, with $Re\ \psi > 0$ leading to a Subcritical Hopf Bifurcation.

This result matches the idea that was suggested in M.d.M.González, M.P.Gualdani and J.Solà-Morales' paper, and takes us one step closer to admitting that claims 3.1.1 and 3.1.2 are valid. Both claims stated the following:

- For $R = R_0$ an Andronov-Hopf type bifurcation occurs, giving rise to periodic oscillations.
- At $R = R_0$ a family of periodic solutions does appear near γ^1.
- For $\phi = \phi_3$ the bifurcation is supercritical, and stable oscillations seem to persist for all $R > R_0$.

Providing that system 2.16 satisfies the conditions of the Hopf Bifurcation Theorem 3.3.1, as seen in Theorems 3.2.1 and 3.2.3, as well as in Figures 4.4 - 4.6, one can admit that for every value of $R > R_0$ there will exist a unique limit cycle and a family of periodic oscillations. Moreover, calculating ψ as in 3.6 and in Appendix B.3, we see that

```
% Choosing phi(w) = tanh(w)/tanh(1) as the nonlinearity
phi = @(w) tanh(w)/tanh(1);
psi1 = eigen_jacobiana21(phi)
psi1 = -27.5709 +87.2966i

% Choosing phi(w) = w as the nonlinearity
phi = @(w) w;
psi2 = eigen_jacobiana21(phi)
psi2 = 1.3400 -23.5090i
```

letting us prove that for $\phi(w) = tanh(w)/tanh(1)$ the bifurcation is supercritical, and on the other hand, for $\phi(w) = w$ the bifurcation will be subcritical, and providing a proof for claims 3.1.1 and 3.1.2.

To sum up, the studied model coming from Mean Field Game theory offers us an interesting price-formation-like behavior for the solutions found for values of $R > 9.36$, which is the range that guarantees a desired and natural instability. Finally, one can say that the initial claims are solved, that this Bifurcation satisfies the Hopf Bifurcation Theorem, and that the critical values of the bifurcation are known, as well as its behaviour, being able to conclude this dissertation.

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Appendix A

Period against R

Even though the main goal of the project was to clarify Claim 3.1.1, there has been additional work on properties of the model.

The main result about this divergent work is how the period of the oscillations in the superitical case, using $\phi = \tanh(x)/\tanh(1)$ as a non-linearity, changes when the constant value R is modified.

The best way to actually see how this phenomena occurs is to calculate the eigenvalues of the linearisation and obtain the period as

$$T = \frac{2\pi}{|Im\ \lambda^*|} \tag{A.1}$$

Where T is the period in this case, and λ^* is the eigenvalue of the linearisation that has the highest real part.

Repeating the process of finding the period for different values of R, for example $R = 15, .., 65$ will give us a good idea of how the period vaies against the R value.

An alternative way to do so would be to plot a solution for T big enough. Then the idea is to approximate the behaviour of one of the points of the discretisation using a least squares method for a trigonometric function. Thus, one will obtain the value of certain parameters for which a trigonometric-like function would approximate our solution obtaining a proper period approximation. The function to minimize is the following one

$$h(t) = \theta_1 + \theta_2 \sin(t\theta_3 + \theta_4) \tag{A.2}$$

and the period in this case is obtained as

$$T = \frac{2\pi}{\theta_3} \tag{A.3}$$

The plotted representation for this last scenario is the following figure

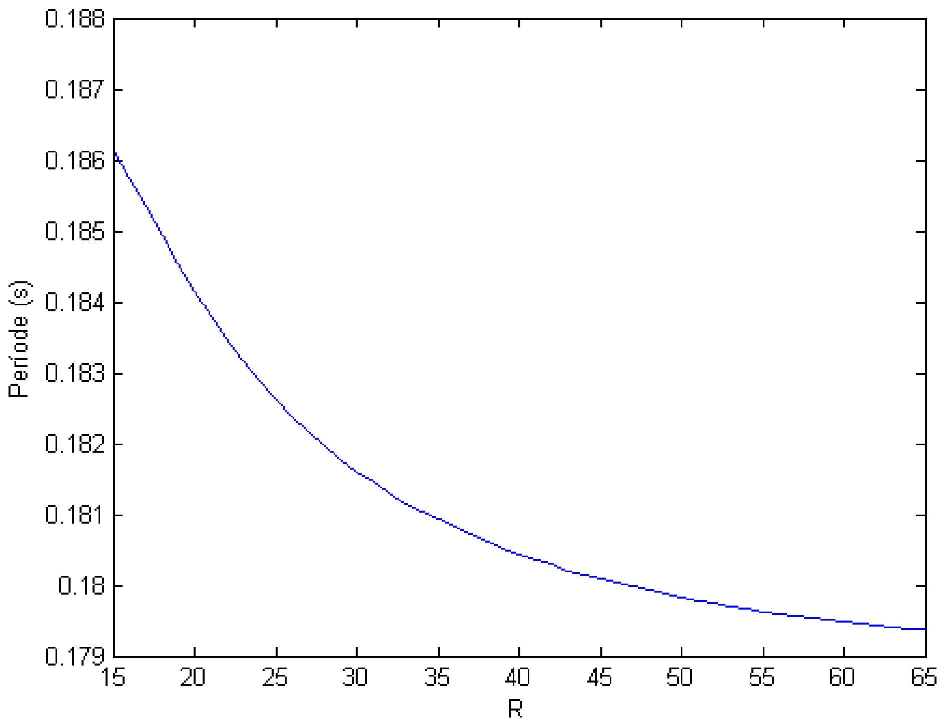

Fig. A.1 Plotted periods for $R \in [15, 65]$ in the supercritical case

This gives us the idea that the radius of the stable limit cycle is reduced as R increases, and for values of R around R_0, the period increases quickly. This fits the idea that a bifurcation *"appears all of a sudden"*, and it is interesting to see that the period seems to stabilise or at least decrease gently as R increases. The behaviour of this relationship around the critical value is left as an open problem for further studies, since the algebra behind it appears to be interesting as a bifurcation problem.

Appendix B

Script

This project has a positive amount of coding and computational work, this is why the most important part of it will be attached in this Appendix.

B.1 Discretisation

```
clear all
% Let's set a = 1, as to simplify the code.
n = 500; % The vector's size will be 2n+1 and  w0 will be at position n+1
H = 2; % Boundary, w_end
h = H/n; % It always has to be he inverse of an integer, 1/k, with k integer
VR = 9.39;% R value of the PDE, seems to start being unstable at R=9.36.
Tf = 10; % Final time

if round(h^-1) == h^-1 & gcd(H,n) == H & n > H
    for r =1:4
        R = VR(r);
        % f = @(x) -sign(x); % Initial condition (1)
        f = @(x) -tanh(5*x)/tanh(10); % Initial condition (2)

        % theta = @(x) sign(x); % Nonlinearity (1)
```

```
% theta = @(x) x; % Nonlinearity (2)
theta = @(x) tanh(x)/tanh(1); % Nonlinearity (3)
X = ([1:2*n+1]-1)*h - n*h;

w = zeros(2*n+1,1);
for i = 1:2*n+1
    w(i) = f(X(i));
end

A1 = -(2/h^2)*eye(2*n+1);

for i=1:2*n
   A1(i,i+1) = 1/(h^2);
   A1(i+1,i) = 1/(h^2);
end

b1 = zeros(2*n+1,1);
b1(1) = 1/(h^2); b1(2*n+1) = -1/(h^2);
% A1x + b1 = w_xx

p =@(x) -((x(n+2)+x(n))/(h^2))/((x(n+2)-x(n))/(2*h))
   ;
A2 = zeros(2*n+1);
for i = 1:2*n
    A2(i,i+1) = 1/(2*h);
    A2(i+1,i) = -1/(2*h);
end

b2 = zeros(2*n+1,1);
b2(1) = -1/(2*h); b2(2*n+1) = -1/(2*h);
% p*(A2x + b2) = p'(t)w_x

auxc = zeros(2*n+1,1); auxc(n+1 -h^-1) = 1/h; auxc(n
   +1 + h^-1) = -1/h;
c =@(x) ((x(n+2)-x(n))/(2*h))*auxc;
% w_x(0,t)[d_-1 -d_1] = c
```

```



            auxd1 = zeros(2*n+1,1); auxd2 = zeros(2*n+1,1);
                auxd1(n+1-h^-1) = 1/h; auxd2(n+1+h^-1) = -1/h;
            d =@(x) R*p(x)*(zeros(2*n+1,1) + theta(x(n+1-h^-1))*
                auxd1 + theta(x(n+1+h^-1))*auxd2);
            % Rp'(t)[operations with theta and delta] = d

            F = @(t,x) (A1*x +b1) + p(x)*(A2*x + b2) -c(x) -d(x)
                ;

            [T,Y] = ode45(F,[0 Tf],w);
            Y = Y';
            [x,y] = size(Y);
            RES = zeros(x+2,y);
            RES(1,:) = ones(1,y); RES(x+2,:) = -ones(1,y);
            RES(2:x+1,1:y) = Y;
            XF = zeros(x+2,1); XF(1) = X(1)-h; XF(x+2) = X(x)+h;
                XF(2:x+1) = X;

            plot(XF,RES(:,y),'-g')
            xlabel(['R = ',num2str(R)])
        end
else
    E = errordlg('h has to be an integer! H < n and gcd(H,n)
        = H','Error');
end
```

B.2 Period against R

```
load('data.m') % Lets us work with a discretisation already
    made
for Ri = 1:length(VR);
    R = VR(Ri);
    F = @(t,x) (A1*x +b1) + p(x)*(A2*x + b2) -c(x) -R*d(x);
    [T,Y] = ode45(F,[0 Tf],w);
    Yf = Y(end,:);
```

```
    [Ta,Ya] = ode45(F,[Tf Tf+epsilon],Yf);

   v = 2; % Chosing a single point to approximate.
   fsol = @(xx) Ya(:,v)- (xx(1) + xx(2)*sin(xx(3)*Ta + xx(4)
      ));
   MM = max(Ya(:,v)); Mm = min(Ya(:,v)); M = mean(Ya(:,v));
   if R>50
     lsqOpts = optimoptions('lsqnonlin', 'TolFun', 1e-14, '
        MaxFunEvals',1000);
     res = lsqnonlin(fsol,[M;(MM-Mm)/2;35;1], [], [],lsqOpts
        );
   else
     res = lsqnonlin(fsol,[M;(MM-Mm)/2;35;1]);
   end
%    figure(i)
%    plot(Ta,Ya(:,v),Ta,Ya(:,v)-fsol(res))
   PERIODE(Ri) = 2*pi/res(3)
end
figure(1+length(VR))
plot(VR, PERIOD)
xlabel('R')
ylabel('Period (s)')
```

B.3 Finding ψ

```
clear all;
load('f_jac.mat');

w = [ones(1,n/2),-[-1:2/n:1],-ones(1,n/2)]; % Stationary
   solution

maxEig =@(R) max(real(eig(f_jac(R,w))));

% Bisection method
start = 1; final = 50;
niter = 200;
for i = 1:niter;
```

```
      m = (start+final)/2;
      res = maxEig(m);
      f0 = maxEig(start);
      ff = maxEig(final);
      if (sign(f0) == sign(res))
          start = m;
      else
          final = m;
      end
end
R_Critica = m;

J = f_jac(R_Critica,w);
JI = inv(J);

[V,Dreal] = eig(J);
Dreal = diag(Dreal);
D = abs(real(Dreal));
[aux,m1] = min(D);
eig1 = Dreal(m1);
D(m1) = inf;
[aux,m2] = min(D);
% Here we find the poistion of the critical eigenvalues inside D.
uaux = V(:,m1);
vaux = V(:,m2);
p = uaux'*vaux;
pc = conj(p);
u = uaux'*sqrt(pc/(norm(p)^2));
v = vaux*sqrt(pc/(norm(p)^2));
% u*v % Here we see how u*v=1.

JIAUX = inv(J-2*eig1*eye(length(u)));
val_critic = 0;
FF = F(x,R_Critica);

```

```
l_total = length(u);
M1 = -ones(l_total,l_total,l_total);
M2 = -ones(l_total,l_total,l_total);
for i = 1:l_total
    f1 = FF(i);
    for q = 1:length(u)
        f2 = FF(q);
        for j = 1:length(v)
            for k = 1:length(v)
                if(M1(q,j,k) == -1)
                    f2_2 = matlabFunction(diff(diff(f2,x(j)),x(k)),'vars',x);
                    f2_2 = f2_2(w);
                    M1(q,j,k) = f2_2;
                    M1(q,k,j) = f2_2;
                end
                f2_2 = M1(q,j,k);

                for l = 1:length(v)
                    if(M2(j,k,l) == -1)
                        f1_1 = matlabFunction(diff(diff(diff(f1,x(j)),x(k)),x(l)),'vars',x);
                        f1_1 = f1_1(w);
                        M2(j,k,l) = f1_1; M2(j,l,k) = f1_1; M2(k,l,j) = f1_1; M2(k,j,l) = f1_1; M2(l,k,j) = f1_1; M2(l,j,k) = f1_1;
                    end
                    f1_1 = M2(j,k,l);

                    if(M1(q,k,l) == -1)
                        f2_1 = matlabFunction(diff(diff(f2,x(k)),x(l)),'vars',x);
                        f2_1 = f2_1(w);
                        M1(q,k,l) = f2_1;
                        M1(q,l,k) = f2_1;
```

```
                    end
                    f2_1 = M1(q,k,l);

                    for m = 1:length(v)
                        if (M1(i,j,m) == -1)
                            f1_2 = matlabFunction(diff(diff(f1,x(j)),x(m)),'vars',x);
                            f1_2 = f1_2(w);
                            M1(i,j,m) = f1_2;
                            M1(i,m,j) = f1_2;
                        end
                        f1_2 = M1(i,m,j);

                        if (M1(i,l,m) == -1)
                            f1_3 = matlabFunction(diff(diff(f1,x(l)),x(m)),'vars',x);
                            f1_3 = f1_3(w);
                            M1(i,l,m) = f1_3;
                            M1(i,m,l) = f1_3;
                        end
                        f1_3 = M1(i,l,m);

                        T = f1_1 - 2*f1_2*JI(m,q)*f2_1 - f1_3*JIAUX(m,q)*f2_2;
                        val_critic = val_critic + u(i)*v(j)*v(k)*v(l)'*T;
                    end
                end
            end
        end
    end
end
PSI = val_critic
```

B.4 Finding ψ for The Van der Pol Oscillator

```
clear all;
```

```
R_Critica = 0;
F = @(x,R) [0,1;-1,R]*x + [0;-x(2)*x(1)^2];
x = sym('x', [2 1]);
w = [0,0];
J = [0,-1;1,0];
JI = inv(J);

[V,Dreal] = eig(J);
Dreal = diag(Dreal);
D = abs(real(Dreal));
[aux,m1] = min(D);
eig1 = Dreal(m1);

u = V(:,1);
v = V(:,2);
% u*v

JIAUX = inv(J-2*eig1*eye(length(u)));
val_critic = 0;
FF = F(x,R_Critica)
pas = 1;
l_total = length(u);
M1 = -ones(l_total,l_total,l_total);
M2 = -ones(l_total,l_total,l_total);
for i = 1:pas:l_total
    f1 = FF(i);
    for q = 1:pas:length(u)
        f2 = FF(q);
        for j = 1:pas:length(v)
            for k = 1:pas:length(v)
                f2_2 = matlabFunction(diff(diff(f2,x(j)),x(k)),'vars',x);
                f2_2 = f2_2(w(1), w(2));
                for l = 1:pas:length(v)
                    f1_1 = matlabFunction(diff(diff(diff(f1,x(j)),x(k)),x(l)),'vars',x);
```

```
                    f1_1 = f1_1(w(1), w(2));
                    f2_1 = matlabFunction(diff(diff(f2,x(k))
                      ,x(1)),'vars',x);
                    f2_1 = f2_1(w(1), w(2));
                    for m = 1:pas:length(v)
                        f1_2 = matlabFunction(diff(diff(f1,x
                          (j)),x(m)),'vars',x);
                        f1_2 = f1_2(w(1), w(2));
                        f1_3 = matlabFunction(diff(diff(f1,x
                          (1)),x(m)),'vars',x);
                        f1_3 = f1_3(w(1), w(2));

                        T = f1_1 - 2*f1_2*JI(m,q)*f2_1 -
                          f1_3*JIAUX(m,q)*f2_2;
                        val_critic = val_critic + u(i)*v(j)*
                          v(k)*v(l)'*T;
                    end
                end
            end
        end
    end
end
val_critic
```